Unit Tangent Vector

**Def.** Let \( C \) be a smooth curve represented by \( \vec{r}(t) \) on an open interval \( I \). The **unit tangent vector** \( \vec{T}(t) \) at \( t \) is defined to be

\[
\vec{T}(t) = \frac{\vec{r}'(t)}{||\vec{r}'(t)||} \quad \text{for } \vec{r}'(t) \neq \vec{0}
\]

**Note:** Recall that a curve is “smooth” on an interval if \( \vec{r}'(t) \) is **continuous** and **nonzero** on the interval. Thus, “smoothness” is sufficient to guarantee that a curve has a unit tangent vector.

**Exercise 1a (Section 12.4 #2):** Determine and sketch the unit tangent vector to the curve \( \vec{r}(t) = t\vec{i} + 2t^2\vec{j} \) at \( t = 1 \).

\[
\vec{r}'(t) = 3t^2\vec{i} + 4t\vec{j}
\]

\[
\vec{T}(t) = \frac{3t^2\vec{i} + 4t\vec{j}}{||3t^2\vec{i} + 4t\vec{j}||}
\]

\[
\vec{T}(1) = \frac{3(1)^2\vec{i} + 4(1)\vec{j}}{\sqrt{9(1)^4 + 16(1)^2}} = \frac{3\vec{i} + 4\vec{j}}{\sqrt{25}} = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j}
\]

**Def.:** The **tangent line to a curve at a point** is the line that passes through the point and that is parallel to the unit tangent vector.

**Exercise 1b:** Determine and sketch the line tangent to the curve at the point corresponding to \( \vec{r}(1) \).

Direction vector: \( \vec{T}(1) = \frac{3}{5}\vec{i} + \frac{4}{5}\vec{j} \)

Point corresponding to \( \vec{r}(1) \): \( (1, 2) \)

Tangent line: \( x = 1 + \frac{3}{5}s \)

\( y = 2 + \frac{4}{5}s \)

\( -\infty < s < \infty \)

or \( \vec{L}(s) = (1 + \frac{3}{5}s)\vec{i} + (2 + \frac{4}{5}s)\vec{j} \)
**Principal Unit Normal Vector**

**Def.:** Let $C$ be a smooth curve represented by $\vec{r}(t)$ on an open interval $I$. If $\vec{T}'(t) \neq \vec{0}$, then the *principal unit normal vector* $\vec{N}(t)$ at $t$ is defined to be

$$\vec{N}(t) = \frac{\vec{T}'(t)}{||\vec{T}'(t)||}$$

**Exercise 1c:** Determine $\vec{N}(t)$ and $\vec{N}(1)$, and sketch $\vec{N}(1)$, for the curve from Exercise 1a.

$\vec{T}'(t) = -\frac{1}{2}(9t^4 + 16t^2)^{3/2} (36t^3 + 32t)(3t^2 \vec{i} + 4t \vec{j}) + (9t^4 + 16t^2)^{3/2} (6t \vec{i} + 4 \vec{j})$

$$= -\frac{18t^5 + 16t}{(9t^4 + 16t^2)^{3/2}} (3t^2 \vec{i} + 4t \vec{j}) + \frac{9t^4 + 16t^2}{(9t^4 + 16t^2)^{3/2}} (6t \vec{i} + 4 \vec{j})$$

To form a common denominator,

$$= \frac{-54t^5 - 48t^3 + 54t^3 + 96t^3}{(9t^4 + 16t^2)^{3/2}} \vec{i} + \frac{-72t^4 - 64t^2 + 36t^4 + 64t^2}{(9t^4 + 16t^2)^{3/2}} \vec{j} = \frac{48t^3}{(9t^4 + 16t^2)^{3/2}} \vec{i} - \frac{36t^4}{(9t^4 + 16t^2)^{3/2}} \vec{j}$$

$$||\vec{T}'(t)|| = \sqrt{\frac{2304t^6}{(9t^4 + 16t^2)^3} + \frac{1296t^8}{(9t^4 + 16t^2)^3}} = \sqrt{\frac{144t^6(16 + 9t^2)}{(9t^4 + 16t^2)^3} + \frac{12t^3 \sqrt{16 + 9t^2}}{(9t^4 + 16t^2)^{3/2}}}$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{||\vec{T}'(t)||} = \frac{(9t^4 + 16t^2)^{3/2}}{12t^3 \sqrt{16 + 9t^2}} \left[ \frac{48t^3}{(9t^4 + 16t^2)^{3/2}} \vec{i} - \frac{36t^4}{(9t^4 + 16t^2)^{3/2}} \vec{j} \right]$$

$$= \frac{4}{\sqrt{16 + 9t^2}} \vec{i} - \frac{3t}{\sqrt{16 + 9t^2}} \vec{j}$$

$$\vec{N}(1) = \frac{4}{\sqrt{16 + 9(1)^2}} \vec{i} - \frac{3(1)}{\sqrt{16 + 9(1)^2}} \vec{j} = \frac{4}{5} \vec{i} - \frac{3}{5} \vec{j}$$
Relationship Between Velocity Vectors and Acceleration Vectors

**Case 1:** If an object is traveling at a constant speed, then the velocity vector at any point is perpendicular to the acceleration vector at that point. This makes sense because if any acceleration were acting either in or against the direction of motion, then the speed would not be constant. We can also prove this assertion using Property 7 on page 2 of Handout 12.2.

**Proof:** “constant speed” \( \implies \|\vec{v}(t)\| = c \) for any \( t \) \( \implies \|\vec{r}'(t)\| = c \) \( \implies (\|\vec{r}'(t)\|)^2 = c^2 \)

\[ \Rightarrow \vec{r}'(t) \cdot \vec{r}''(t) = k \]

Case 2: If an object is traveling at a variable speed, then the velocity vector at any point is not necessarily perpendicular to the acceleration vector at that point. For example, the acceleration vector of a projectile always points down, regardless of the direction of motion.

**Tangential and Normal Components of Acceleration**

**Reminder:** Recall that two vectors determine a plane. So, if \( \vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 \), then \( \vec{w} \) lies in the plane determined by \( \vec{v}_1 \) and \( \vec{v}_2 \).

**Note:** \( \vec{w} \) is said to be a linear combination of \( \vec{v}_1 \) and \( \vec{v}_2 \).

**Theorem:** If \( \vec{r}(t) \) is the position vector for a smooth curve \( C \) and if \( \vec{N}(t) \) exists, then the acceleration vector \( \vec{a}(t) \) lies in the plane determined by \( \vec{T}(t) \) and \( \vec{N}(t) \).

**Proof:** \( \vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\vec{v}(t)}{\|\vec{v}(t)\|} \Rightarrow \vec{v}(t) = \|\vec{v}(t)\| \vec{T}(t) \)

**Differentiate:** \( \vec{v}'(t) = \left(\|\vec{v}(t)\|\right)' \vec{T}(t) + \|\vec{v}(t)\| \vec{T}'(t) \)

\[ \vec{a}(t) = \frac{\left(\|\vec{v}(t)\|\right)'}{\|\vec{T}'(t)\|} \vec{T}(t) + \|\vec{v}(t)\| \vec{T}'(t) \cdot \frac{\|\vec{T}'(t)\|}{\|\vec{T}'(t)\|} \vec{N}(t) \]

\[ = \left(\|\vec{v}(t)\|\right)' \vec{T}(t) + \|\vec{v}(t)\| \vec{T}'(t) \vec{N}(t) \]

\[ \vec{a}_T \quad \vec{a}_N \]
**Theorem:** If \( \vec{r}(t) \) is the position vector for a smooth curve \( C \) and if \( \vec{N}(t) \) exists, then the tangential and normal components of acceleration as are follows:

\[
\begin{align*}
a_T &= \left[ \| \vec{v}(t) \| \right]' \\
&= \vec{a}(t) \cdot \vec{T}(t) \\
&= \frac{\vec{v}(t) \cdot \vec{a}(t)}{\| \vec{v}(t) \|} \\
a_T &= \text{tangential component of acceleration} \\
&= \text{the part of acceleration that acts in the line of motion}
\end{align*}
\]

\[
\begin{align*}
a_N &= \| \vec{v}(t) \| \| \vec{T}'(t) \| \\
&= \sqrt{\| \vec{a}(t) \|^2 - a_T^2(t)} \\
&= \vec{a}(t) \cdot \vec{N}(t) \\
&= \frac{\| \vec{v}(t) \times \vec{a}(t) \|}{\| \vec{v}(t) \|} \\
a_N &= \text{the normal component of acceleration} \\
&= \text{the part of acceleration that acts perpendicular to the line of motion}
\end{align*}
\]
Exercise 2: Let \( \vec{r}(t) = 4t \hat{i} + 3\cos(t) \hat{j} + 3\sin(t) \hat{k} \). Determine \( \vec{T}(t) \), \( \vec{N}(t) \), \( a_r \), and \( a_N \) at \( t = \frac{\pi}{2} \).

\[
\vec{r}'(t) = 4\hat{i} - 3\sin(t) \hat{j} + 3\cos(t) \hat{k}
\]

\[
\|\vec{r}'(t)\| = \|\vec{v}(t)\| = \sqrt{16 + 9\sin^2(t) + 9\cos^2(t)} = \sqrt{16 + 9(\sin^2(t) + \cos^2(t))} = \sqrt{16 + 9(1)} = \sqrt{25} = 5
\]

So, \( \vec{T}(t) = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{4\hat{i} - 3\sin(t) \hat{j} + 3\cos(t) \hat{k}}{5} \)

Thus, \( \vec{T}\left(\frac{\pi}{2}\right) = \frac{4}{5}\hat{i} - \frac{3}{5}\hat{j} + 0\hat{k} \)

\[
\vec{r}'(t) = 0\hat{i} - \frac{3}{5}\cos(t) \hat{j} - \frac{3}{5}\sin(t) \hat{k}
\]

Hence, \( \|\vec{r}'(t)\| = \sqrt{0^2 + \frac{9}{25}\cos^2(t) + \frac{9}{25}\sin^2(t)} = \sqrt{\frac{9}{25}(\cos^2(t) + \sin^2(t))} = \frac{3}{5} \)

So, \( \vec{N}(t) = \frac{\vec{r}''(t)}{\|\vec{r}'(t)\|} = \frac{0\hat{i} - \frac{3}{5}\cos(t) \hat{j} - \frac{3}{5}\sin(t) \hat{k}}{\frac{3}{5}} \)

\( = \frac{0\hat{i} - \cos(t) \hat{j} - \sin(t) \hat{k}}{\frac{3}{5}} \)

Thus, \( \vec{N}\left(\frac{\pi}{2}\right) = 0\hat{i} + 0\hat{j} - \hat{k} \)

\( a_r = \left[ \|\vec{v}(t)\| \right] = \left[ \begin{array}{c} 5 \end{array} \right] = 0 \)

Thus, \( a_r\left(\frac{\pi}{2}\right) = 0 \).

\( a_N = \|\vec{v}(t)\| \|\vec{r}'(t)\| \)

\( = 5 \cdot \frac{3}{5} = 3 \)

Thus, \( a_N\left(\frac{\pi}{2}\right) = 3 \).